YENSEN TENGSIZLIGI VA UNING TENGSIZLIKLARNI ISBOTLASHGA TATBIQLARI

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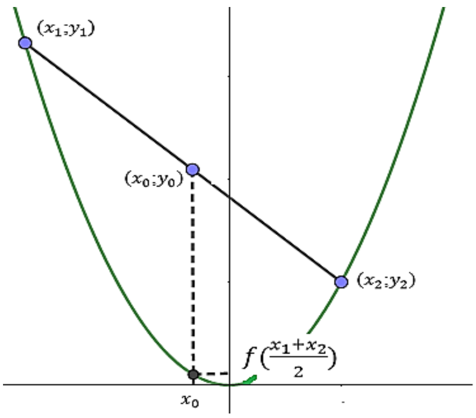
Annotatsiya: Ushbu maqolada tengsizliklarni isbotlashda va boshqa shu turdagi olimpiada masalalarni yechishda yaxshi vosita Yensen tengsizligi tavsifi va uni masalalar yechishga qo‘llashga doir namunalar keltirilgan.
Kalit so‘zlar: Qavariq funksiya,botiq funksiya,Yensen tengsizligi,Minkovskiy tengsizligi,hosila.

Quyida tengsizliklarni isbotlashda keng qo‘llaniladigan muhim bo‘lgan Yensen tengsizligi va undan foydalanib klassik tengsizliklar (Koshi-Shvarts-Bunyakovskiy,Gyolder,Minkovskiy) isbotlarini va boshqa tengsizliklarni isbotlashga tadbiqlarini ko‘rib o‘tamiz.

Yensen tengsizligini qo‘llashda asosiy vosita – qavariq funksiya ta’rifi va qavariqlik kriteriyasini keltiramiz.

$y = x^2$ funksiya va unga tegishli $(x_1; y_1)$ va $(x_2; y_2)$ nuqta berilgan bo‘lsin. (1-rasm) Ma’lumki y_1 va y_2 nuqtalar o‘rtasining koordinatasi $y_0 = \frac{f(x_1)+f(x_2)}{2}$ ga teng bo‘ladi. Grafikdan ko‘rish mumkinki $f(x_0) \le y_0$. Bundan

$f(\frac{x_1+x_2}{2}) \le \frac{f(x_1)+f(x_2)}{2}$ (1) tengsizlikka ega bo‘lamiz.



1-rasm.

(1) ifoda *Yensen*[1] *tengsizligi* deyiladi va biz quyida uning umumiy holda isbotini keltiramiz.

Ta'rif 1. $f:[a,b]\to R$ funksiya uchun ixtiyoriy $x,y\in[a,b]$ sonlar va $\alpha_1+\alpha_2=1$ shartni qanoatlantiruvchi ixtiyoriy α_1,α_2 sonlar uchun

$$f(\alpha_1 x+\alpha_2 y)\le \alpha_1 f(x)+\alpha_2 f(y) \quad (2)$$

munosabat o'rinli bo'lsa, $f(x)$ funksiya $[a,b]$ oraliqda *quyidan qavariq funksiya* (teskari tengsizlik uchun *yuqoridan qavariq*) deyiladi.

Endi umumiy holda Yensen tengsizligini keltiramiz.

Teorema 1(Yensen tengsizligi). $y=f(x)$ biror oraliqda quyidan qavariq va $x_1,x_2,\ldots,x_n$ shu oraliqdagi sonlar bo'lsin. $\alpha_1+\alpha_2+\cdots+\alpha_n=1$ shartni qanoatlantiruvchi $\alpha_1,\alpha_2,\ldots,\alpha_n$ sonlar uchun

$$f(\alpha_1 x_1+\cdots+\alpha_n x_n)\le \alpha_1 f(x_1)+\cdots+\alpha_n f(x_n) \quad (3)$$

tengsizlik o'rinli.

Ushbu teorema isbotini keltirib o'tmaymiz.

Misol 1. Yensen tengsizligini qo'llab Koshi-Shvarts tengsizligini isbotlaymiz.

Ixtiyoriy musbat $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$ sonlar uchun

$$(a_1^2+a_2^2+\cdots+a_n^2)\cdot(b_1^2+b_2^2+\cdots+b_n^2)\ge(a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

tengsizlikni isbotlang.

Isbot. Ma'lumki $y=x^2$ funksiya quyidan qavariq. Bu funksiya uchun Yensen tengsizligini yozamiz:

$$\left(\frac{m_1x_1+\cdots+m_nx_n}{m_1+\cdots+m_n}\right)^2\le\frac{x_1^2m_1+\cdots+x_n^2m_n}{m_1+\cdots+m_n}\quad (m_i>0).$$

Bu ifodani soddalashtirib:

$(x_1^2m_1+\cdots+x_n^2m_n)\cdot(m_1+\cdots+m_n)\ge(m_1x_1+\cdots+m_nx_n)^2$ ni hosil qilamiz.

Oxirgi ifodada $m_i=b_i^2$, $x_i=\frac{a_i}{b_i}$ almashtirish olib talab qilingan tengsizlikni hosil qilamiz.

Yensen tengsizligini samarali qo'llay olish uchun funksiyani qavariq ekanligini bilish kriteriyasi bilish muhimdir.Buning uchun funksiya qavariqlik shartini ifodalovchi teoremani isbotsiz keltiramiz.

Teorema 2. $(a;b)$ oraliqda uzluksiz va ikkinchi tartibli hosilaga ega $f:(a;b)\to\mathbb{R}$ shu oraliqda quyidan(yuqoridan) qavariq bo'lishi uchun shu oraliqda $f''(x)>0$ $(f''(x)<0)$ tengsizlik bajarilishi zarur va yetarli.

Endi bevosita amaliy masalalar tahlillariga o'tsak bo'ladi.

Misol 2. (O‘rta qiymatlar haqidagi Koshi tengsizligi). Nomanfiy $a_i, i = 1..n$ sonlar uchun

$$\frac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \cdot \ldots \cdot a_n} \quad (4)$$

tengsizlik o‘rinli.

Isbot. $y = lnx$ funksiya qaramiz. Ma’lumki $y'' = -\frac{1}{x^2} < 0$. Bundan berilgan funksiya yuqoridan qavariq bo‘ladi. (4) tengsizlik $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$ ko‘rinishda bo‘ladi. U holda $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$ bo‘lib, $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \ln(\frac{1}{n}\alpha_1 + \cdots + \frac{1}{n}\alpha_n)$ (a) va $\alpha_1 f(x_1) + \cdots + \alpha_n f(x_n) = \frac{1}{n}ln\alpha_1 + \cdots + \frac{1}{n}ln\alpha_n = \frac{1}{n}ln(\alpha_1 \cdot \ldots \cdot \alpha_n) = \ln(\alpha_1 \cdot \ldots \cdot \alpha_n)^{\frac{1}{n}}$ (b). (a) va (b) ifodalarga Yensen tengsizligini qo‘llab (6) tengsizlikni hosil qilamiz.

Misol 3. α, β, γ uchburchak burchaklari bo‘lsin.

$$sin\alpha sin\beta sin\gamma \leq \frac{3\sqrt{3}}{8}$$

tengsizlikni isbotlang.

Yechish. $GM \leq AM$ ga ko‘ra $\sqrt[3]{sin\alpha sin\beta sin\gamma} \leq \frac{sin\alpha+sin\beta+sin\gamma}{3} \Longrightarrow sin\alpha sin\beta sin\gamma \leq \left(\frac{sin\alpha+sin\beta+sin\gamma}{3}\right)^3$.

Ma’lumki $\alpha, \beta, \gamma \in (0;\pi)$ va $f(x) = sinx$ funksiya $(0;\pi)$ oraliqda yuqoridan qavariq.U holda Yensen tengsizligini yuqoridan qavariq funksiya uchun qo‘llasak:

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}, \Rightarrow \frac{1}{3}sin\alpha + \frac{1}{3}sin\beta + \frac{1}{3}sin\gamma \leq sin\frac{\alpha+\beta+\gamma}{3} = \frac{\sqrt{3}}{2}.$$

Oxirgi ifodadan : $\sqrt[3]{sin\alpha sin\beta sin\gamma} \leq \frac{\sqrt{3}}{2} \Rightarrow sin\alpha sin\beta sin\gamma \leq \frac{3\sqrt{3}}{8}$ ni hosil qilamiz.

Misol 4. $a, b, c \in \mathbb{R}^+$ bo‘lsin. Quyidagi tengsizlikni isbotlang:

$$9(a^3 + b^3 + c^3) \geq (a + b + c)^3$$

Yechish. Yensen tengsizligini muvaffaqiyatli qo‘llay olish uchun mos funksiya to‘g’ri tanlash muhimdir. Bu tengsizliklar uchun $y = x^3$ funksiyani qarasak u $(0;\infty)$ da quyidan qavariq funksiyadir. Haqiqatdan ham $f''(x) = 6 > 0$.U holda Yensen tengsizligidan quydagiga ega bo‘lamiz:

$$\left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3} \Longrightarrow 9(a^3 + b^3 + c^3) \geq (a + b + c)^3.$$

Misol 5. Tengsizlikni isbotlang: $\sqrt[n]{a_1 \cdot \ldots \cdot a_n} + \sqrt[n]{b_1 \cdot \ldots \cdot b_n} \leq \sqrt[n]{(a_1 + b_1) \cdot \ldots \cdot (a_n + b_n)}$ (5)

Isbot. Tengsizlikni ikkala tomonini $\sqrt[n]{a_1 \cdot \ldots \cdot a_n}$ ga bo‘lib yuborib:

$1+\left(\frac{b_1}{a_1}\right)^{\frac{1}{n}}\cdot\ldots\cdot\left(\frac{b_n}{a_n}\right)^{\frac{1}{n}}\le\left(1+\frac{b_1}{a_1}\right)^{\frac{1}{n}}\cdot\ldots\cdot\left(1+\frac{b_n}{a_n}\right)^{\frac{1}{n}}$ ni hosil qilamiz. Quyidagicha belgilash kiritamiz: $x_i=ln\frac{b_i}{a_i}$. U holda oxirgi ifodani

$1+e^{\frac{1}{n}\sum x_i}\le\prod(1+x_i)^{\frac{1}{n}}$ (6) ko'rinishda yozish mumkin.(Bunda $\sum x_i=x_1+\cdots+x_n$), $\prod(1+x_i)=(1+x_1)\ldots(1+x_n)$).

Yensen tengsizligini qo'llash uchun funksiyani $f(x)=\ln(1+e^x)$ kabi tanlaymiz. Ma'lumki:$f''(x)=\frac{e^x}{(1+e^x)^2}>0$. Demak bu funksiya quyidan qavariq ekan. (8) ifodani ikki tomonini logarifmlab: $\ln(1+e^{\frac{1}{n}\sum x_i})\le\frac{1}{n}\sum\ln(1+x_i)$ talab qilingan tasdiqqa ega bo'lamiz. (5) tengsizlik *Minorskiy tengsizligi* deb nomlanadi.

Misol 6. Agar $x+y+z=1$ *bo'lsa,* $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\ge 64xyz$ *tengsizlikni isbotlang.*

Isbot. $y=ln(1+\frac{1}{t})$ funksiyani qaraymiz. U uchun $y''=\frac{2t+1}{(t(t+1))^2}>0$. Demak, bu funksiya quyidan qavariq. U holda Yensen tengsizligi uchun $\alpha_1=\alpha_2=\alpha_3=\frac{1}{3}$ kabi tanlaymiz. U holda $f(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)=\ln\left(1+\frac{1}{\frac{1}{3}(x+y+z)}\right)=\ln\left(1+\frac{3}{x+y+z}\right)=ln4$. (a)

$\alpha_1f(x_1)+\alpha_2f(x_2)+\alpha_3f(x_3)=\frac{1}{3}\ln\left(1+\frac{1}{x}\right)+\frac{1}{3}\ln\left(1+\frac{1}{y}\right)+\frac{1}{3}\ln\left(1+\frac{1}{z}\right)=\frac{1}{3}\ln\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)$ *(b)*

(a) va (b) ifodalarga Yensen tengsizligini qo'llab talab qilingan tasdiqqa ega bo'lamiz.

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